

Classical Higgs fields

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Abstract

We consider classical gauge theory on a principal bundle $P \rightarrow X$ in a case of spontaneous symmetry breaking characterized by the reduction of a structure group G of $P \rightarrow X$ to its closed subgroup H . This reduction is ensured by the existence of global sections of the quotient bundle $P/H \rightarrow X$ treated as classical Higgs fields. Matter fields with an exact symmetry group H in such gauge theory are considered in the pairs with Higgs fields, and they are represented by sections of a composite bundle $Y \rightarrow P/H \rightarrow X$, where $Y \rightarrow P/H$ is a fiber bundle associated to a principal bundle $P \rightarrow P/H$ with a structure group H . A key point is that a composite bundle $Y \rightarrow X$ is proved to be associated to a principal G -bundle $P \rightarrow X$. Therefore, though matter fields possess an exact symmetry group $H \subset G$, their gauge G -invariant theory in the presence of Higgs fields can be developed. Its gauge invariant Lagrangian factorizes through the vertical covariant differential determined by a connection on a principal H -bundle $P \rightarrow P/H$. In a case of the Cartan decomposition of a Lie algebra of G , this connection can be expressed in terms of a connection on a principal bundle $P \rightarrow X$, i.e., gauge potentials for a group of broken symmetries G .

1 Introduction

In general, classical gauge theory comprises fields of three types: gauge potentials, matter fields, and classical Higgs fields. The latter are attributes of gauge theory in a case of spontaneous symmetry breaking.

Spontaneous symmetry breaking is a quantum phenomenon when automorphism of a quantum algebra need not preserve its vacuum state [1, 2]. In this case, we have inequivalent vacuum states of a quantum system which are classical objects. For instance, spontaneous symmetry breaking in Standard Model of particle physics is ensured by the existence of a constant vacuum Higgs field [3, 4].

Classical field theory adequately is formulated in terms of Lagrangian theory on smooth fiber bundles whose sections are classical fields [5, 6]. Correspondingly, classical gauge theory is classical field theory on principal and associated bundles

[6, 7]. In gauge theory on a principal bundle $P \rightarrow X$, spontaneous symmetry breaking pertains to reducing its structure Lie group G to a closed subgroup H of exact symmetries [8, 9, 10, 11]. Such a reduction is possible iff the quotient fiber bundle $P/H \rightarrow X$ admits global sections h (Theorem 1). These sections can be interpreted as classical Higgs fields [8, 9, 11, 12, 13]. They parameterize the principal reduced subbundles P^h (with a structure group H) of a principal bundle P . These subbundles are inequivalent (Remark 3) and need not be isomorphic (Theorems 4 – 5).

If a structure group G of a principal bundle $P \rightarrow X$ is reduced to a closed subgroup H , then in the framework of this gauge theory, we can consider matter fields with an exact symmetry group H . They are described by sections s_h of fiber bundles Y^h which possess a typical fiber V endowed with the left action of a group H , and which are associated to principal reduced subbundles $P^h \subset P$. Because the subbundles P^h for different h fails to be equivalent, such matter fields can enter the theory only in a pair with a certain Higgs field h . A problem of describing the set of all pairs (s_h, h) of matter and Higgs fields thus arises. These pairs are represented by sections of the composite bundle $Y \rightarrow P/H \rightarrow X$ (21), where $Y \rightarrow P/H$ is a fiber bundle with a structure group H and a typical fiber V , and this fiber bundle is associated to a principal H -bundle $P \rightarrow P/H$ (Section 5). The geometry of such composite bundles has been studied in [6, 12, 13]. A key observation is that, for any section h of the quotient bundle $P/H \rightarrow X$, the pull-back h^*Y of a fiber bundle $Y \rightarrow P/H$ is a subbundle Y^h of $Y \rightarrow X$ which is associated to a principal reduced subbundle $P^h \subset P$ with a structure group H . Its sections s_h correspond to matter fields in the presence of a background Higgs field h .

Following [13], we here prove that a composite bundle $Y \rightarrow X$ is a P -associated bundle with a structure group G (Theorem 14). This allows describing matter fields with an exact symmetry group H in terms of gauge theory on a principal bundle P (Section 6). A key point is that a Lagrangian of these matter fields factorizes through the vertical covariant differential \tilde{D} (29), determined by an H -connection on a fiber bundle $Y \rightarrow P/H$. The restriction A_h of this connection to a subbundle $Y^h \subset Y$ then becomes an H -connection on this subbundle (Proposition 12), and the restriction Y^h of the vertical covariant differential \tilde{D} does the differential covariant with respect to a connection A_h (Proposition 13).

A problem however is that a connection on a fiber bundle $Y \rightarrow P/H$ is not a dynamical variable in gauge theory. We therefore assume that the Lie algebra of a group G admits the Cartan decomposition (18). In this case, any G -connection on a principal bundle $P \rightarrow X$ yields an H -connection on any reduced subbundle P^h (Theorem 8) and, therefore, induces a desired H -connection on a fiber bundle $Y \rightarrow P/H$ (Theorem 15). On the configuration space (40), this results in gauge theory of gauge potentials of a group G , of matter fields with an exact symmetry

subgroup $H \subset G$, and of classical Higgs fields.

For instance, this is the case of gravitation theory on an oriented 4-dimensional manifold X . It is formulated as gauge theory with spontaneous symmetry breaking on the principal bundle LX of linear frames tangent to X with a structure group $GL^+(4, \mathbb{R})$ reduced to the Lorentz group $SO(1, 3)$ [8, 14, 15, 16]. Global sections of the corresponding quotient bundle $LX/SO(1, 3)$ are pseudo-Riemannian metrics on the manifold X , which are identified with gravitational fields in General Relativity. The underlying physical reason of this spontaneous symmetry breaking is both the geometric Equivalence principle and the existence of Dirac spinor fields with the Lorentz spin group of symmetries.

2 Gauge theory on principal bundles

We consider smooth fiber bundles (of class C^∞). Smooth manifolds throughout are assumed to be separable, locally compact, countable at infinity, paracompact topological spaces.

As already mentioned, we formulate classical gauge theory on a principal bundle

$$\pi_P : P \rightarrow X \quad (1)$$

over an n -dimensional manifold X with a structure Lie group G acting on P on the right fiberwise freely and transitively [6, 7]. For brevity, we call P the principal G -bundle. Its atlas

$$\Psi_P = \{(U_\alpha, z_\alpha), \varrho_{\alpha\beta}\} \quad (2)$$

is defined by a family of local sections z_α with G -valued transition functions $\varrho_{\alpha\beta}$, such that $z_\beta(x) = z_\alpha(x)\varrho_{\alpha\beta}(x)$, $x \in U_\alpha \cap U_\beta$.

Because G acts on P on the right, one considers the quotient bundles

$$T_GP = TP/G, \quad V_GP = VP/G \quad (3)$$

over X . A typical fiber of the bundle $V_GP \rightarrow X$ is the right Lie algebra \mathfrak{g}_r of a group G with the basis $\{e_p\}$ on which G acts by the adjoint representation. Sections of the fiber bundles $T_GP \rightarrow X$ and $V_GP \rightarrow X$ (3) are G -invariant vector fields and vertical G -invariant vector fields on P , respectively.

In a general setting, a connection on a principal bundle $P \rightarrow X$ is defined as a section of a jet bundle $J^1P \rightarrow P$, where J^1P is the jet manifold of a fiber bundle $P \rightarrow X$ [7, 17]. Because connections on a principal bundle are assumed to be equivariant with respect to the structure group action (for brevity, we call them G -connections), they are identified to global sections of the quotient bundle

$$C = J^1P/G \rightarrow X, \quad (4)$$

coordinated by (x^μ, a_μ^p) . This is an affine bundle modelled over a vector bundle $T^*X \otimes_X V_G P$. Because of the canonical embedding

$$C \xrightarrow{X} dx^\mu \otimes (\partial_\mu + a_\mu^p e_p) \in T^*X \otimes_X T_G P,$$

G -connections on P can also be represented in terms of $T_G P$ -valued forms

$$A = dx^\lambda \otimes (\partial_\lambda + A_\lambda^p e_p). \quad (5)$$

Let V be a manifold admitting a left action of a structure group G of the principal bundle P (1). a fiber bundle associated to P with a typical fiber V is then defined as the quotient space

$$Y = (P \times V)/G, \quad (p, v)/G = (pg, g^{-1}v)/G, \quad g \in G. \quad (6)$$

For brevity, we call it a P -associated bundle.

Every atlas Ψ_P (2) of a principal bundle P determines an atlas

$$\Psi_Y = \{(U_\alpha, \psi_\alpha)\}, \quad \psi_\alpha(x) : (z_\alpha(x), v)/G \rightarrow v, \quad (7)$$

of the associated bundle Y (6), and endows Y with fiberwise coordinates (x^λ, y^i) .

Every G -connection A (5) on a principal bundle P yields a connection

$$A = dx^\lambda \otimes (\partial_\lambda + A_\lambda^p I_p^i \partial_i) \quad (8)$$

on an associated bundle Y , where $\{I_p\}$ is a representation of the Lie algebra \mathfrak{g}_r in a typical fiber V .

3 Reduced structures and Higgs fields

Let H , $\dim H > 0$, be a closed subgroup of a structure group G (see Remark 1 below). There is a composite fiber bundle

$$P \rightarrow P/H \rightarrow X, \quad (9)$$

where

$$P_\Sigma = P \xrightarrow{\pi_{P\Sigma}} P/H \quad (10)$$

is a principal bundle with a structure group H and

$$\Sigma = P/H \xrightarrow{\pi_{\Sigma X}} X \quad (11)$$

is a P -associated bundle with a typical fiber G/H on which a structure group G acts on the left.

Remark 1. A closed subgroup H of a Lie group G is a Lie group. We consider the quotient space G/H of a group G with respect to the right action of H on G . One can show that

$$\pi_{GH} : G \rightarrow G/H \quad (12)$$

is a principal bundle with a structure group H [18]. In particular, if H is a maximal compact subgroup of G , then the quotient space G/H is diffeomorphic to \mathbb{R}^m , and the fiber bundle (12) is trivial.

A structure Lie group G of a principal bundle P is said to be reduced to its closed subgroup H if the following equivalent conditions are satisfied:

- a principal bundle P admits the atlas (2) with H -valued transition functions $\varrho_{\alpha\beta}$;
- there exists a principal reduced subbundle P_H of a principal bundle P with a structure group H .

Indeed, if $P_H \subset P$ is a reduced subbundle, then its atlas (2), generated by local sections z_α also is an atlas of a principal bundle P with H -valued transition functions. Conversely, let (2) be an atlas of a principal bundle P with H -valued transition functions $\varrho_{\alpha\beta}$. For any $x \in U_\alpha \subset X$, we define a submanifold $z_\alpha(x)H \subset P_x$. These submanifolds constitute an H -subbundle of P because $z_\alpha(x)H = z_\beta(x)H\varrho_{\beta\alpha}(x)$ on intersections $U_\alpha \cap U_\beta$.

Remark 2. Principal reduced H -subbundles of a principal G -bundle sometimes are called the G -structures [11, 19, 20, 21]. In [19, 21], only reduced structures of a principal bundle LX of linear frames in the tangent bundle TX of a manifold X were considered, and the isomorphism class of these structures was confined to holonomy automorphisms of LX , i.e., to functorial lift onto LX of diffeomorphisms of a base X . A notion of the G -structure was extended to an arbitrary fiber bundle in [20], where it was interpreted as the Klein-Chern geometry. In a case where the Lie algebra of a group G admits the Cartan decomposition (18), the G -structure is said to be reduced [22], and it manifests several additional features (Theorem 8).

A key point is the following [23].

THEOREM 1. *There is one-to-one correspondence*

$$P^h = \pi_{P\Sigma}^{-1}(h(X)) \quad (13)$$

between the principal reduced H -subbundles $i_h : P^h \rightarrow P$ of a principal bundle P and the global sections h of the quotient bundle $P/H \rightarrow X$ (11).

The relation (13) implies that a principal reduced H -subbundle P^h is the restriction h^*P_Σ of the principal H -bundle P_Σ (10) to a submanifold $h(X) \subset \Sigma$. At the same time, every atlas Ψ_h of a principal bundle P^h generated by a family of its local sections also is an atlas of a principal G -bundle P and an atlas of the P -associated bundle $\Sigma \rightarrow X$ (11) with H -valued transition functions. Relative to an atlas Ψ_h of a fiber bundle Σ , a global section h of this bundle takes its values in the center of the quotient space G/H .

As already mentioned, we treat global sections of the quotient bundle $P/H \rightarrow X$ as classical Higgs fields in classical gauge theory [6, 11, 13].

Reducing a structure group is not always possible. In particular, in an above mentioned case of gauge gravitation theory, it occurs on noncompact manifolds X and on compact manifolds with the zero Euler characteristic. We note the following fact [18].

THEOREM 2. *A bundle $Y \rightarrow X$ whose typical fiber is diffeomorphic to a manifold \mathbb{R}^m always admits a global section, and every section of it over a closed submanifold of a base X can be extended to the global one.*

COROLLARY 3. *A structure group G of a principal bundle P is reducible to a closed subgroup H if the quotient space G/H is diffeomorphic to an Euclidean space \mathbb{R}^m .*

In particular, we always have a reduction of a Lie structure group G to its maximal compact subgroup H (see Remark 1). This is the case $G = GL(m, \mathbb{C})$, $H = U(m)$ and $G = GL(n, \mathbb{R})$, $H = O(n)$, which are essential for applications.

Let us note that different principal H -subbundles P^h and $P^{h'}$ of a principal G -bundle P need not be mutually isomorphic.

THEOREM 4. *If the quotient space G/H is diffeomorphic to an Euclidean space \mathbb{R}^m , then all principal reduced H -subbundles of a principal G -bundle P are mutually isomorphic [18].*

THEOREM 5. *Let a Lie structure group G of a principal bundle P be reduced to its closed subgroup H . We have the following statements:*

- *Every vertical automorphism Φ of a principal bundle P maps its principal reduced H -subbundle P^h to an isomorphic principal reduced H -subbundle $P^{h'} = \Phi(P^h)$.*

- *Conversely, let two reduced subbundles P^h and $P^{h'}$ of a principal bundle $P \rightarrow X$ be mutually isomorphic, and let $\Phi : P^h \rightarrow P^{h'}$ be their isomorphism over X . Then it is extended to an automorphism of a principal bundle P .*

Proof: Let

$$\Psi^h = \{(U_\alpha, z_\alpha^h), \varrho_{\alpha\beta}^h\} \quad (14)$$

be an atlas of a principal reduced subbundle P^h , where z_α^h are local sections of $P^h \rightarrow X$ and $\varrho_{\alpha\beta}^h$ are transition functions. Given a vertical automorphism Φ of a fiber bundle P , its subbundle $P^{h'} = \Phi(P^h)$ is provided with an atlas

$$\Psi^{h'} = \{(U_\alpha, z_\alpha^{h'}), \varrho_{\alpha\beta}^{h'}\}, \quad (15)$$

determined by local sections $z_\alpha^{h'} = \Phi \circ z_\alpha^h$. Then one can easily obtain

$$\varrho_{\alpha\beta}^{h'}(x) = \varrho_{\alpha\beta}^h(x), \quad x \in U_\alpha \cap U_\beta, \quad (16)$$

i.e., transition functions of the atlas (15) take their values in a subgroup H . Conversely, every automorphism $(\Phi, \text{Id } X)$ of principal reduced subbundles P^h and $P^{h'}$ of a principal bundle P determines an H -equivariant G -valued function f on P^h by the relation $pf(p) = \Phi(p)$, $p \in P^h$. Its extension to a G -equivariant function on P is defined as

$$f(pg) = g^{-1}f(p)g, \quad p \in P^h, \quad g \in G.$$

The relation $\Phi_P(p) = pf(p)$, $p \in P$, then defines a vertical automorphism Φ_P of a principal bundle P whose restriction to P^h coincides with Φ . \square

Remark 3. In Theorem 5, we can regard a principal G -bundle P endowed with the atlas Ψ^h (14) as a P^h -associated fiber bundle with a structure group H acting on its typical fiber G on the left. Correspondingly, being equipped with the atlas $\Psi^{h'}$ (15), a principal bundle P is a $P^{h'}$ -associated H -bundle. However, the H -bundles (P, Ψ^h) and $(P, \Psi^{h'})$ are not equivalent because their atlases Ψ^h and $\Psi^{h'}$ fail to be equivalent. Indeed, the union of these atlases is an atlas

$$\Psi = \{(U_\alpha, z_\alpha^h, z_\alpha^{h'}), \varrho_{\alpha\beta}^h, \varrho_{\alpha\beta}^{h'}, \varrho_{\alpha\alpha} = f(z_\alpha)\}$$

with transition functions

$$\varrho_{\alpha\alpha}(x) = f(z_\alpha(x)), \quad z_\alpha^{h'}(x) = z_\alpha^h(x)\varrho_{\alpha\alpha}(x) = (\Phi_P \circ z_\alpha^h)(x),$$

between the corresponding charts (U_α, z_α^h) and $(U_\alpha, z_\alpha^{h'})$ of atlases Ψ^h and $\Psi^{h'}$, respectively. However transition functions $\varrho_{\alpha\alpha}$ are not H -valued. At the same time, the equality (16) implies that transition functions of both atlases constitute the same cocycle. Hence, the H -bundles (P, Ψ^h) and $(P, \Psi^{h'})$ are associated. Owing to an isomorphism $\Phi : P^h \rightarrow P^{h'}$, we can write

$$P = (P^h \times G)/H = (P^{h'} \times G)/H, \quad (p \times g)/H = (\Phi(p) \times f^{-1}(p)g)/H.$$

For any $\rho \in H$, we then obtain

$$\begin{aligned} (p\rho, g)/H &= (\Phi(p)\rho, f^{-1}(p)g)/H = (\Phi(p), \rho f^{-1}(p)g)/H \\ &= (\Phi(p), f^{-1}(p)\rho'g)/H, \end{aligned}$$

where

$$\rho' = f(p)\rho f^{-1}(p). \quad (17)$$

Hence, we can treat $(P, \Psi^{h'})$ as a P^h -associated bundle with the same typical fiber G as for (P, Ψ^h) , but the action $g \rightarrow \rho'g$ (17) of a structure group H on a typical fiber G of the bundle $(P, \Psi^{h'})$ is not equivalent to its action $g \rightarrow \rho g$ on a typical fiber G of the bundle (P, Ψ^h) , because they have different orbits in G .

4 Reduction of connections

We present compatibility conditions for connections on a principal bundle with its reduced structures [7, 23].

THEOREM 6. *Since connections on a principal bundle are equivariant, every H -connection A_h on an H -subbundle P^h of a principal G -bundle P is extendible to a G -connection on P .*

THEOREM 7. *Conversely, the connection A (5) on a principal G -bundle P is reducible to an H -connection on a principal reduced H -subbundle P^h of P iff the corresponding global section h of the quotient bundle $P/H \rightarrow X$ associated to P is an integral section of the associated connection A (8) on $P/H \rightarrow X$, i.e., $D^A h = 0$, where D^A is the covariant differential determined by this connection A .*

In particular, a connection on P always is reducible to a connection on P^h under the following condition [6, 23].

THEOREM 8. *Let the Lie algebra \mathfrak{g} of a Lie group G be a direct sum*

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{f} \tag{18}$$

of the Lie algebra \mathfrak{h} of a Lie group H and its complement \mathfrak{f} such that we have the condition $[\mathfrak{h}, \mathfrak{f}] \subset \mathfrak{f}$. Let A be a G -connection on a principal bundle P . We consider a principal reduced bundle P^h with an atlas Ψ_h , which also is an atlas of a principal bundle P . Then the pull-back $\overline{A}_h = h^ A_{\mathfrak{h}}$ on P^h of the \mathfrak{h} -valued constituent $A_{\mathfrak{h}}$ of the connection form A (5) written with respect to an atlas Ψ_h is an H -connection on P^h .*

In this case, matter fields with an exact symmetry group H can be written in the presence of gauge fields with a larger group G of spontaneously broken symmetries (Section 6).

In particular, the decomposition (18) occurs if H is the Cartan subgroup of G , and we therefore call this the Cartan decomposition.

For instance, in gauge gravitation theory on the principal frame bundle LX , the Lie algebras $\mathfrak{g} = gl(4, \mathbb{R})$ and $\mathfrak{h} = so(1, 3)$ satisfy the condition (18), and for a given pseudo-Riemannian metric h , a general linear connection can be decomposed into the sum of Christoffel symbols, the contorsion tensor, and the nonmetricity tensor. Its first two terms constitute the Lorentzian connection on a reduced $SO(1,3)$ -subbundle $L^h X \subset LX$, which allows describing Dirac spinor fields in gravitation theory in the presence of a general linear connection [14, 15, 16].

We can also generate connections on principal reduced subbundles in a different way.

Let $P \rightarrow X$ be a principal bundle. For a morphism of manifolds $\phi : X' \rightarrow X$, the pull-back bundle $\phi^* P \rightarrow X'$ is a principal bundle with the same structure

group as of P . If A is a connection on a principal bundle P , then the pull-back connection ϕ^*A on ϕ^*P is a connection on it as on a principal bundle [7]. We hence obtain a result important in what follows [6, 7].

THEOREM 9. *We consider the composite bundle (9). Let A_Σ be a connection on the principal H -bundle $P \rightarrow \Sigma$ (10). Then for any principal reduced H -bundle $i_h : P^h \rightarrow P$, the pull-back connection $i_h^*A_\Sigma$ on P^h is an H -connection on this bundle*

Let us note that, as already mentioned in Introduction, a Lagrangian of matter fields factorizes through the vertical covariant differential determined just by a connection on a fiber bundle $P \rightarrow P/H$ (Section 6).

5 Associated bundles and matter fields

By virtue of Theorem 1, there is one-to-one correspondence between the principal reduced H -subbundles P^h of a principal bundle P and the Higgs fields h . Given such a subbundle P^h , let us consider an associated bundle

$$Y^h = (P^h \times V)/H \quad (19)$$

with a typical fiber V admitting the left action of an exact symmetry group H . Its sections s_h describe matter fields in the presence of a Higgs field h and an H -connection A_h on a principal bundle P^h .

Different fiber bundles Y^h and $Y^{h' \neq h}$ (19) are mutually related as follows. If the principal reduced H -subbundles P^h and $P^{h'}$ of a principal G -bundle P are isomorphic by virtue of Theorem 5, then the P^h -associated bundle Y^h (19) also is associated as

$$Y^h = (\Phi(p) \times V)/H \quad (20)$$

to a subbundle $P^{h'}$. If its typical fiber V admits the action of a whole group G , then the P^h -associated bundle Y^h (19) also is P -associated,

$$Y^h = (P^h \times V)/H = (P \times V)/G.$$

Such P -associated bundles are isomorphic as G -bundles, but not equivalent as H -bundles, because transition functions between their atlases are not H -valued (see Remark 3).

For example, in gauge gravitation theory on a manifold X , the tangent bundle TX treated for a given pseudo-Riemannian metric h as a L^hX -associated bundle is a fibering into copies of the Minkowski space M^hX . However for different pseudo-Riemannian metrics h and h' , the fiber bundles M^hX and $M^{h'}X$ are not equivalent; in particular, representations of their elements in terms of γ -matrices are not equivalent [16, 24].

Because different fiber bundles Y^h and $Y^{h' \neq h}$ are not equivalent and need not be isomorphic, one must consider a V -valued matter field only in pair with a certain Higgs field. We therefore encounter a problem of description of a set of all pairs (s_h, h) of matter and Higgs fields.

To describe matter fields in the presence of different Higgs fields, we consider the composite bundle (9) and a composite bundle

$$Y \xrightarrow{\pi_{Y\Sigma}} \Sigma \xrightarrow{\pi_{\Sigma X}} X, \quad (21)$$

where $Y \rightarrow \Sigma$ is a P_Σ -associated bundle

$$Y = (P \times V)/H \quad (22)$$

with a structure group H . For a given global section h of the fiber bundle $\Sigma \rightarrow X$ (11) and for the corresponding principal reduced H -bundle $P^h = h^*P$, the fiber bundle (19) associated to P^h is the restriction

$$Y^h = h^*Y = (h^*P \times V)/H \quad (23)$$

of a fiber bundle $Y \rightarrow \Sigma$ to $h(X) \subset \Sigma$.

One can then prove the following statements [6, 12, 13].

PROPOSITION 10. *Every global section s_h of the fiber bundle Y^h (23) is a global section of the composite fiber bundle (21) projected onto a section $h = \pi_{Y\Sigma} \circ s$ of a fiber bundle $\Sigma \rightarrow X$. Conversely, any global section s of the composite fiber bundle $Y \rightarrow X$ (21), when projected onto a section $h = \pi_{Y\Sigma} \circ s$ of a fiber bundle $\Sigma \rightarrow X$, takes its values in the subbundle Y^hY (23). Hence, there is one-to-one correspondence between the sections of the fiber bundle Y^h (19) and those of the composite bundle (21) that cover h .*

PROPOSITION 11. *An atlas*

$$\Psi_{P\Sigma} = \{(U_{\Sigma\iota}, z_\iota), \varrho_{\iota\kappa}\} \quad (24)$$

of a principal H -bundle $P \rightarrow \Sigma$ and correspondingly of an associated bundle $Y \rightarrow \Sigma$ defines an atlas

$$\Psi^h = \{(\pi_{P\Sigma}(U_{\Sigma\iota}), z_\iota \circ h), \varrho_{\iota\kappa} \circ h\} \quad (25)$$

of a reduced H -subbundle P^h and hence of an associated bundle Y^h , which also is an atlas of a principal bundle P with H -valued transition functions.

Given an atlas Ψ_P of a principal bundle P , which determines the atlas of the associated bundle $\Sigma \rightarrow X$ (11), and an atlas $\Psi_{Y\Sigma}$ of a fiber bundle $Y \rightarrow \Sigma$, we can endow the composite fiber bundle $Y \rightarrow X$ (21) with the corresponding coordinate

system $(x^\lambda, \sigma^m, y^i)$, where (σ^m) are fiberwise coordinates on $\Sigma \rightarrow X$ and (y^i) are those on $Y \rightarrow \Sigma$.

PROPOSITION 12. *Let*

$$A_\Sigma = dx^\lambda \otimes (\partial_\lambda + \mathcal{A}_\lambda^a e_a) + d\sigma^m \otimes (\partial_m + \mathcal{A}_m^a e_a) \quad (26)$$

be a principal connection on a principal H -bundle $P \rightarrow \Sigma$, and let

$$A_{Y\Sigma} = dx^\lambda \otimes (\partial_\lambda + \mathcal{A}_\lambda^a(x^\mu, \sigma^k) I_a^i \partial_i) + d\sigma^m \otimes (\partial_m + \mathcal{A}_m^a(x^\mu, \sigma^k) I_a^i \partial_i) \quad (27)$$

be an associated connection on $Y \rightarrow \Sigma$, where $\{I_a\}$ is a representation of the right Lie algebra \mathfrak{h}_r of a group H in V . Then, for any H -subbundle $Y^h \rightarrow X$ of a composite bundle $Y \rightarrow X$, the pull-back connection

$$A_h = h^* A_{Y\Sigma} = dx^\lambda \otimes [\partial_\lambda + (\mathcal{A}_m^a(x^\mu, h^k) \partial_\lambda h^m + \mathcal{A}_\lambda^a(x^\mu, h^k)) I_a^i \partial_i], \quad (28)$$

on Y^h is associated to the pull-back connection $h^ A_\Sigma$ on a principal reduced H -subbundle P^h in Theorem 9.*

Every connection A_Σ (26) on a fiber bundle $Y \rightarrow \Sigma$ determines the first-order differential operator

$$\tilde{D} : J^1 Y \rightarrow T^* X \otimes_{V_Y} V_\Sigma Y, \quad \tilde{D} = dx^\lambda \otimes (y_\lambda^i - \mathcal{A}_\lambda^i - \mathcal{A}_m^i \sigma_\lambda^m) \partial_i, \quad (29)$$

acting on a composite bundle $Y \rightarrow X$, where $V_\Sigma Y$ is the vertical tangent bundle to a fiber bundle $Y \rightarrow \Sigma$. It is called the vertical covariant differential, and has the following important property.

PROPOSITION 13. *For any section h of a fiber bundle $\Sigma \rightarrow X$, the restriction of the vertical differential \tilde{D} (29) on the fiber bundle Y^h (23) coincides with the covariant differential D^{A_h} on Y^h with respect to the pull-back connection A_h (28).*

We thus find that those are sections of the composite bundle $Y \rightarrow X$ (21) that describe the pairs (s_h, h) of matter and Higgs fields in classical gauge theory with spontaneous symmetry breaking.

The following fact is essential when constructing gauge theory with spontaneous symmetry breaking [13, 25].

THEOREM 14. *The composite bundle $Y \rightarrow X$ (21) is a P -associated bundle whose structure group is G and whose typical fiber is an H -bundle*

$$W = (G \times V)/H, \quad (30)$$

associated to a principal H -bundle $G \rightarrow G/H$ (12).

Proof: Let us represent a fiber bundle $P \rightarrow X$ as a P -associated bundle

$$P = (P \times G)/G, \quad (pg', g) = (p, g'g), \quad p \in P, \quad g, g' \in G,$$

whose typical fiber is a group space of G on which a group G acts by left multiplications. We can then represent the quotient (22) in a form

$$Y = (P \times (G \times V)/H)/G, \\ (pg', (g\rho, v)) = (pg', (g, \rho v)) = (p, g'(g, \rho v)) = (p, (g'g, \rho v)).$$

Therefore, Y (22) is a P -associated bundle with the typical fiber W (30) on which the structure group G acts according to the law

$$g' : (G \times V)/H \rightarrow (g'G \times V)/H. \quad (31)$$

This is the so-called induced representation of a group G by its subgroup H [26]. Given an atlas $\{(U_a, z_a)\}$ of a principal H -bundle $G \rightarrow G/H$, the induced representation (31) takes a form

$$g' : (\sigma, v) = (z_a(\sigma), v)/H \rightarrow (\sigma', v') = (g'z_a(\sigma), v)/H = \\ (z_b(\pi_{GH}(g'z_a(\sigma)))\rho', v)/H = (z_b(\pi_{GH}(g'z_a(\sigma))), \rho'v)/H, \\ \rho' = z_b^{-1}(\pi_{GH}(g'z_a(\sigma)))g'z_a(\sigma) \in H, \quad \sigma \in U_a, \quad \pi_{GH}(g'z_a(\sigma)) \in U_b.$$

For example, if H is the Cartan subgroup of G , then the induced representation (31) is a known nonlinear realization of the group G [6, 27, 28]. \square

6 Lagrangian of matter fields

Propositions 12 – 13 and Theorem 14 imply the following peculiarity of formulating Lagrangian gauge theory with spontaneous symmetry breaking [13, 29].

Let $P \rightarrow X$ be a principal bundle whose structure group G is reduced to a closed subgroup H . Let Y be the P_Σ -associated bundle (22). A total configuration space of gauge theory of G -connections on P in the presence of matter and Higgs fields is

$$J^1C \times_X J^1Y, \quad (32)$$

where C is the quotient bundle (4) and J^1Y is the manifold of jets of a fiber bundle $Y \rightarrow X$. A total Lagrangian on the configuration space (32) is a sum

$$L_{\text{tot}} = L_A + L_m + L_\sigma \quad (33)$$

of a gauge field Lagrangian L_A , a matter field Lagrangian L_m , and a Higgs field Lagrangian L_σ .

Because we do not specify gauge and Higgs fields and because their Lagrangians can take rather different forms depending on a model, for instance, in gauge gravitation theory and in Yang–Mills theory, we here consider only a matter field Lagrangian L_m . By virtue of Proposition 13, it factorizes as

$$L_m : J^1 C \times_{\substack{X \\ Y}} J^1 Y \xrightarrow{\tilde{D}} T^* X \otimes_{\substack{Y \\ X}} V_{\Sigma} Y \rightarrow \wedge^n T^* X \quad (34)$$

through the vertical differential \tilde{D} (29). Moreover, we can demonstrate that such a factorization is a necessary condition for the gauge invariance of L_m under automorphisms of a principal G -bundle $P \rightarrow X$ [6].

However a problem is that the connection A_{Σ} (26) on a fiber bundle $Y \rightarrow P/H$, which determines \tilde{D} , is not a dynamical variable in gauge theory. We therefore assume that the Lie algebra of a group G admits the Cartan decomposition (18). In this case, any G -connection A on a principal bundle $P \rightarrow X$ determines an H -connection \overline{A}_h on every reduced subbundle P^h (Theorem 8). We can then prove the following theorem [13, 29].

THEOREM 15. *We have the H -connection A_{Σ} (26) on a fiber bundle $Y \rightarrow P/H$ whose restriction $A_h = h^* A_{\Sigma}$ to the P^h -associated bundle Y^h coincides with an H -connection \overline{A}_h generated on P^h by a connection A on a principal G -bundle $P \rightarrow X$.*

Proof: Let a principal reduced subbundle $P^h \subset P$, be given, and let \overline{A}_h be an H -connection on P^h in Theorem 8 generated by the G -connection A on a principal bundle $P \rightarrow X$. By virtue of Theorem 6, we can extend this connection to a G -connection on P for which h is an integral section of the associated connection

$$\overline{A}_h = dx^{\lambda} \otimes (\partial_{\lambda} + A_{\lambda}^p J_p^m \partial_m)$$

on a P -associated bundle $\Sigma \rightarrow X$. With respect to the atlas Ψ^h (14) of a principal bundle P with H -valued transition functions, a Higgs field h takes values in the center of a homogeneous space G/H , and a connection \overline{A}_h is

$$\overline{A}_h = dx^{\lambda} \otimes (\partial_{\lambda} + A_{\lambda}^a e_a). \quad (35)$$

We then obtain

$$A = \overline{A}_h + \Theta = dx^{\lambda} \otimes (\partial_{\lambda} + A_{\lambda}^a e_a) + \Theta_{\lambda}^b dx^{\lambda} \otimes e_b, \quad (36)$$

where $\{e_a\}$ is a basis for the right Lie algebra \mathfrak{h}_r and $\{e_b\}$ is a basis of its complement \mathfrak{f}_r . The decomposition (36) with respect to an arbitrary atlas of a principal bundle P takes a form

$$A = \overline{A}_h + \Theta, \quad \Theta = \Theta_{\lambda}^p dx^{\lambda} \otimes e_p,$$

and satisfies the relation $\Theta_\lambda^p J_p^m = D_\lambda^A h^m$, where D_λ^A are the covariant derivatives with respect to the associated connection A on a fiber bundle $\Sigma \rightarrow X$. Let us consider the covariant differential

$$D = D_\lambda^m dx^\lambda \otimes \partial_m = (\sigma_\lambda^m - A_\lambda^p J_p^m) dx^\lambda \otimes \partial_m$$

with respect to the associated connection A on $\Sigma \rightarrow X$. We can represent this differential as a $V\Sigma$ -valued form on the jet manifold $J^1\Sigma$ of a fiber bundle $\Sigma \rightarrow X$. Because the decomposition (36) holds for any section h of a fiber bundle $\Sigma \rightarrow X$, we obtain a V_GP -valued form $\Theta = \Theta_\lambda^p dx^\lambda \otimes e_p$ on $J^1\Sigma$, which satisfies the equation

$$\Theta_\lambda^p J_p^m = D_\lambda^m. \quad (37)$$

As a result, we obtain a V_GP -valued form

$$A_H = dx^\lambda \otimes (\partial_\lambda + (A_\lambda^p - \Theta_\lambda^p) e_p)$$

on $J^1\Sigma$ whose restriction to every submanifold $J^1h(X) \subset J^1\Sigma$ is the connection \overline{A}_h (35) written with respect to the atlas Ψ^h (25). Because the decomposition (36) holds, the equation (37) admits a solution for any G -connection A . We therefore have the V_GP -valued form

$$A_H = dx^\lambda \otimes (\partial_\lambda + (a_\lambda^p - \Theta_\lambda^p) e_p) \quad (38)$$

on the product $J^1\Sigma \times_X J^1C$ such that for any connection A and for any Higgs field h , the restriction of A_H (38) to

$$J^1h(X) \times A(X) \subset J^1\Sigma \times_X J^1C$$

is the connection \overline{A}_h (35), written with respect to the atlas Ψ^h (25). Now let A_Σ (26) be a connection on a principal H -bundle $P \rightarrow \Sigma$. This connection determines a $V_\Sigma Y$ -valued form

$$\tilde{D} = dx^\lambda \otimes (y_\lambda^i - (\mathcal{A}_m^a \sigma_\lambda^m + \mathcal{A}_\lambda^a) I_a^i) \partial_i \quad (39)$$

(the covariant differential (29)) on the configuration space (32). We now assume that, for a given connection A on a principal G -bundle $P \rightarrow X$, the pull-back connection $A_h = h^* A_{Y\Sigma}$ (28) on Y^h coincides with \overline{A}_h (35) for any $h \in \Sigma(X)$. By virtue of Proposition 13, we can then define components of the form (39) as follows. For a given point

$$(x^\lambda, a_\mu^r, a_{\lambda\mu}^r, \sigma^m, \sigma_\lambda^m, y^i, y_\lambda^i) \in J^1C \times_X J^1Y, \quad (40)$$

let h be a section of a fiber bundle $\Sigma \rightarrow X$ whose jet $j_x^1 h$ in $x \in X$ is $(\sigma^m, \sigma_\lambda^m)$, i.e.,

$$h^m(x) = \sigma^m, \quad \partial_\lambda h^m(x) = \sigma_\lambda^m.$$

Let the connection bundle C and the Lie algebra fiber bundle V_GP be endowed with atlases associated to the atlas Ψ^h (25). We can then write

$$A_h = \overline{A}_h, \quad \mathcal{A}_m^a \sigma_\lambda^m + \mathcal{A}_\lambda^a = a_\lambda^a - \Theta_\lambda^a. \quad (41)$$

These equations for the functions \mathcal{A}_m^a and \mathcal{A}_λ^a at the point (40) have a solution because Θ_λ^a are affine functions in the jet coordinates σ_λ^m . \square

Having the solution of the equation (41), we substitute it in the covariant differential \tilde{D} (39) requiring that the matter field Lagrangian factorizes in the form (34) through the form \tilde{D} (39), called the universal covariant differential determined by a G -connection A on a principal bundle P . As a result, we obtain gauge theory of gauge potentials of a group G , of matter fields with a symmetry subgroup $H \subset G$ and of classical Higgs fields on the configuration space (40).

As mentioned above, an example of a classical Higgs field is a metric gravitational field in gauge gravitation theory on natural fiber bundles with the spontaneous symmetry breaking due to the existence of Dirac spinor fields with the Lorentz spin group of symmetries or by the geometric equivalence principle [8, 14, 15, 16]. Describing spinor fields in terms of the composite bundle (21), we obtain their Lagrangian (34) in the presence of a general linear connection; this Lagrangian is invariant under general covariance transformations [6, 16, 30].

In a more general form, classical Higgs fields also were considered in theory of spinor fields on the so-called gauge-natural fiber bundles [31].

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